MA1002: Linear Algebra

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Text books

1 Linear Algebra, Kenneth Hoffman and Ray Kunze,

Prentice-Hall, Second Edition (available online).

2 Topics in Algebra, I. N. Herstein, Wiley.

3 Linear Algebra and its Applications, Gilbert Strang, 4th edition. 4 Introduction to Linear Algebra, Krishnamurthi (BITS Pilani).

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Evaluation Scheme (Tentative)

| Quiz 1  Quiz 2  End Semester Examination | 25  25  50 |
| --- | --- |

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Applications of Linear Algebra

• To summarize and manipulate data (Machine Learning, Image Processing)

• To model satellites, jet engines (Eigen values)

• Many more!

• Prepare a detailed report of an application of linear algebra tools in CSE/ECE/ME and Design

You have first assignment !

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Mathematical Structures

1 Field (Members are called scalars)

2 Vector Space (Members are called vectors)

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Field Definition

A non empty set *F* together with operations

(i)+ : *F × F → F* (addition) and

(ii) *·* : *F × F → F* (multiplication)

is said to be a field if the following axioms and properties are satisfied.

Closure axiom: For every *a, b ∈ F* =*⇒*

*a* + *b ∈ F* and *a · b ∈ F*

Associative axiom: For all *a, b, c ∈ F* =*⇒*

*a* + (*b* + *c*) = (*a* + *b*) + *c* and

*a ·* (*b · c*) = (*a · b*) *· c*

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Field Contd. Identity axiom: There exist elements 0*,* 1 *∈ F* such that

*a* + 0 = 0 + *a* = *a, ∀a ∈ F*

*a ·* 1 = 1 *· a* = *a*, *∀a ∈ F*

Inverse axiom:(i) For every *a ∈ F*, there exists *b ∈ F* such that

*a* + *b* = *b* + *a* = 0

(*b* is called additive inverse of *a*)

and (ii) for every *a ∈ F − {*0*}*, there exists *c ∈ F − {*0*}* such that

*a · c* = *c · a* = 1

(*c* is called multiplicative inverse of *a*)

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Field Contd.

Commutative Property:

*a* + *b* = *b* + *a ∀a, b ∈ F* and

*a · b* = *b · a ∀a, b ∈ F*

Distributive Property: For every *a, b, c ∈ F*

*a ·* (*b* + *c*) = *a · b* + *a · c* and

(*a* + *b*) *· c* = *a · c* + *b · c*

Notation: A field *F* with respect to operations +*, ·* is usually denoted as (*F,* +*, ·*)

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Field Examples

N = *{*1*,* 2*,* 3*, . . . , }* - set of all natural numbers

Is (N*,* +*, ·*) a field?

Closure axiom:

We know that

for all *a, b ∈* N, *a* + *b* and *a · b* are also elements of N

Associative axiom:

Associative property is true for N for both + and *·*

Identity axiom:

0 *∈/* N and hence identity axiom is not satisfied

Thus (N*,* +*, ·*) is not a field.

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Field Examples

Z = *{. . . , −*2*, −*1*,* 0*,* 1*,* 2*, . . .}* - set of all integers

Is (Z*,* +*, ·*) a field?

Closure axiom: For all *a, b ∈* Z, *a* + *b* and *a · b* are also elements of Z

Associative axiom: Associative property is true for Z for both + and *·*

Identity axiom: There exists 0*,* 1 *∈* Z such that *a* + 0 = 0 + *a* = *a* and *a ·* 1 = 1 *· a* = *a*, *∀a ∈* Z

Inverse axiom: For every *a ∈* Z, there exists *−a ∈* Z such that *a* + (*−a*) = *−a* + *a* = 0. But for 2 *∈* Z there doesnot exist a multiplicative inverse in Z

Thus (Z*,* +*, ·*) is not a field

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Field Examples

Z2 = *{*0*,* 1*}* - congruence class modulo 2.

Is (Z2*,* +*, ·*) a field?

Closure axiom:

0 + 0 = 0*,* 1 + 0 = 1*,* 0 + 1 = 1*,* 1 + 1 = 0

0 *·* 0 = 0*,* 1 *·* 0 = 0*,* 0 *·* 1 = 0*,* 1 *·* 1 = 1

Thus, the closure axiom is true.

Associative axiom:

From closure axiom we can see that associative axiom also holds true.

Identity axiom:

0 *∈* Z2 is the additive identity and 1 *∈* Z2 is the multiplicative identity.

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Inverse axiom:

Additive inverse of 0 is 0 and 1 is 1.

Multiplicative inverse of 1 is 1.

Commutative property:

From closure axiom, we can see that addition and multiplication are commutative in Z2

Distributive property: For every *a, b ∈* Z2

(*a* + *b*) *·* 0 = 0 = *a ·* 0 + *b ·* 0

(*a* + *b*) *·* 1 = *a* + *b* = *a ·* 1 + *b ·* 1

Thus distributive law holds.

Therefore, (Z2*,* +*, ·*) is a field.

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Field Examples

R - Set of real numbers.

Is (R*,* +*, ·*) a field?

Closure axiom:

Addition of two real numbers is a real number. Similarly, Multiplication of two real numbers is a real number.

Thus, the closure axiom is true.

Associative axiom:

Addition and Multiplication are always associative in R.

Identity axiom:

0 *∈* R is the additive identity and 1 *∈* R is the multiplicative identity. 12

Inverse axiom:

Additive inverse of an element *a ∈* R is *−a*.

Multiplicative inverse of an element *a ∈* R *− {*0*}* is 1*a*. Commutative property:

It is true that for every *a, b ∈* R

*a* + *b* = *b* + *a* and *a · b* = *b · a*

Distributive property:

Distributive property is true for real numbers with respect to + and *·* Therefore, (R*,* +*, ·*) is a field.

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Questions

Q- Set of all rational numbers

C- Set of all complex numbers

1 Is (Q*,* +*, ·*) a field?

2 Is (C*,* +*, ·*) a field?

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Linear Equations

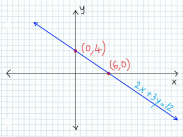
Equation of a line :

*y* = *mx* + *c*

*ax* + *by* = *c*

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Example of a line



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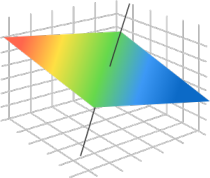
Equation of a plane :

*z* = *ax* + *by* + *c*

*Ax* + *By* + *Cz* = *D*

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Example of a plane



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Systems of linear equations

*A*11*x*1 + *A*12*x*2 + *. . .* + *A*1*nxn* = *b*1

*A*21*x*1 + *A*22*x*2 + *. . .* + *A*2*nxn* = *b*2

: : :

*Am*1*x*1 + *Am*2*x*2 + *. . .* + *Amnxn* = *bm*

• *m* equations

• *n* varaibles (*x*1*, x*2*, . . . , xn*)

• *Aij , bi ∈ F* (*F* is a field)

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Matrix form





*A*11 *A*12 *. . . A*1*n*



*A*21 *A*22 *. . . A*2*n*

*. . . . . . . . . . . .*

*Am*1 *Am*2 *. . . Amn*



*x*1

*x*2

*. . .*

*xn*



=



*b*1

*b*2

*. . .*

*bm*

 

*AX* = *B*

*A* = [*Aij*]*m×n* , 1 *≤ i ≤ m*, 1 *≤ j ≤ n*

*X* = [*xj*]*n×*1, *B* = [*bi*]*m×*1

The solution set of the linear system *AX* = *B* is

*S* = *{X ∈ Rn*: *AX* = *B}*

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Problem 1

Solve the following system of linear equations

4*x − y* = 5

2*x* + *y* = 7

Matrix form

"

4 *−*1 2 1

# "

*x*

*y*

#

=

"

5

7

#

Solve graphically (sketch it!)

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Problem 1 (solution)

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Problem 1

Solve the following system of linear equations

4*x − y* = 5

2*x* + *y* = 7

Matrix form

"

4 *−*1 2 1

# "

*x*

*y*

#

=

"

5

7

#

Solve graphically (sketch it!)

Solution *S* = *{*(2*,* 3)*}*

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Our objective is to design an efficient machinery to solve *AX* = *B*

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Solution of Problem 1 through elementary row operations

4*x − y* = 5 (*Eq*1)

2*x* + *y* = 7 (*Eq*2)

Let us employ a high school technique

Multiply (Eq2) by 2

4*x − y* = 5 (*Eq*1)

4*x* + 2*y* = 14 (*Eq*2)

(Eq2)-(Eq1) =*⇒*

4*x − y* = 5 (*Eq*1)

0*x* + 3*y* = 9 (*Eq*2)

We have three equivalent systems say red, blue and green 25

contd.

Let us express red system in augmented matrix form

"

4 *−*1 5 2 1 7

#

Multiply second row by 2 (*R*2 *←−* 2 *× R*2)

"

4 *−*1 5 2 1 7

#

*∼*

"

4 *−*1 5 4 2 14

#

Substract first row from second row (*R*2 *←− R*2 *− R*1)

"

4 *−*1 5 2 1 7

#

*∼*

"

4 *−*1 5 4 2 14

#

*∼*

"

4 *−*1 5 0 3 9

#

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Contd.

"

4 *−*1 5

2 1 7

*R*2 *←−* 13*R*2 =*⇒*

#

*∼*

"

4 *−*1 5 4 2 14

"

#

*∼*

#

"

4 *−*1 5 0 3 9

#

*R*1 *←−* 14*R*1 =*⇒*

*∼ ∼*

"

4 *−*1 5 0 1 3

1 *−*14540 1 3

#

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Contd.

*R*1 *←−* 14*R*1 =*⇒*

"

*∼*

*R*1 *←− R*1 +14*R*2 =*⇒* "

*∼*

1 *−*14540 1 3

1 0 2 0 1 3

#

#

Let us write it in the system of equations form

*x* + 0*y* = 2

0*x* + *y* = 3 28

Salient points

• Multiplying an equation by a non-zero scalar preserves the solution space (*Ri ←− cRi, c ̸*= 0)

• Replacing *ith* equation by sum of *ith* equation and constant multiple of *jth* equation preserves the solution space.

(*Ri ←− Ri* + *cRj*)

• Interchanging two equations preserves the solution space. (*Ri ←→ Rj*)

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Problem 2

Solve the following system of linear equations. 3*x −* 2*y* = *−*6*, x* + 2*y* = *−*10

Solve graphically !

Augmented matrix is

"

3 *−*2 *−*6 1 2 *−*10

#

Interchange first and second rows (*R*1 *←→ R*2)

*∼*

"

1 2 *−*10 3 *−*2 *−*6

#

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Problem 2 contd.

*R*2 *←− R*2 *−* 3*R*1 =*⇒ ∼*

*R*2 *←− −*18*R*2

*∼*

"

1 2 *−*10 0 *−*8 24

"

1 2 *−*10

0 1 *−*3

#

#

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Problem 2 contd.

*R*1 *←− R*1 *−* 2*R*2 =*⇒*

*∼*

"

1 0 *−*4 0 1 *−*3

#

*x* = *−*4*, y* = *−*3

Solution *S* = *{*(*−*4*, −*3)*}*

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Problem 3

Solve the system of linear equations *x* + *y* = 2*,* 2*x* + 2*y* = 5

Solve graphically!

Augmented matrix is

"

1 1 2 2 2 5

"

#

(*R*2 *←− R*2 *−* 2*R*1) #

*∼*

1 1 2 0 0 1

Second row is 0*x* + 0*y* = 1. No solution

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Problem 4

Solve the system

*x* + 2*y* = 5*,* 2*x* + 4*y* = 10

Solve graphically! Augmented matrix is

"

1 2 5 2 4 10

"

#

(*R*2 *←− R*2 *−* 2*R*1) #

*∼*

1 2 5 0 0 0

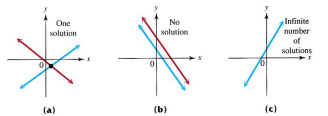
=*⇒ x* + 2*y* = 5

Let *y* = *c* =*⇒ x* = 5 *−* 2*c*.

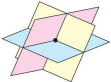
The solution *S* = *{*(5 *−* 2*c, c*) : *c ∈* R*}*

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two dimensional problem and possible solutions

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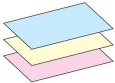
3-dimensional problem with a unique solution



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3-dimensional problem with no solutions





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3-dimensional problem with infinite number of solutions 



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Linear combination of equations

*A*11*x*1 + *A*12*x*2 + *A*13*x*3 = *b*1 (1)

*A*21*x*1 + *A*22*x*2 + *A*23*x*3 = *b*2 (2)

Consider *c*1(1) + *c*2(2)( a linear combination) =*⇒*

*c*1 (*A*11*x*1 + *A*12*x*2 + *A*13*x*3) + *c*2 (*A*21*x*1 + *A*22*x*2 + *A*23*x*3) = *c*1*b*1 + *c*2*b*2 (3)

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Suppose that *x*1 = *a, x*2 = *b, x*3 = *c* is solution of (1) and (2) Show that above solution is also a solution of (3).

Consider L.H.S. of (3),

*c*1 (*A*11*a* + *A*12*b* + *A*13*c*) + *c*2 (*A*21*a* + *A*22*b* + *A*23*c*)

= *c*1*b*1 + *c*2*b*2

So *x*1 = *a, x*2 = *b, x*3 = *c* is solution of (3)

Converse need not be true (Try !)

Note: If *X∗*is a solution of *k* linear equations, then *X∗*is also a solution of a linear combination of those *k* equations.

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Equivalent systems

We say two sytems are equivalent if each equation in each system is a linear combination of equations in the other system.

Why do we focus on equivalent systems?

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Theorem 1: Equivalent systems of linear equations have exactly same solutions.

Proof:

Let (*A*) and (*B*) be two equivalent systems with solution sets *SA* and *SB* respectively. Prove that *SA* = *SB* .

Let *X ∈ SA*. =*⇒ X* satisfies every equation in (*A*), and every equation in (*B*) is a linear combination of equations in (*A*). =*⇒ X* satisfies every equation in (*B*). =*⇒ X ∈ SB*

Hence *SA ⊆ SB*

Similarly, *SB ⊆ SA*

=*⇒ SA* = *SB*

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Problem

Show that the following systems of linear equations are equivalent.

*x − y* = 0

2*x* + *y* = 0*} − − − − −* (*I*)

3*x* + *y* = 0

*x* + *y* = 0*} − − − − −* (*II*)

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Solution

3*x* + *y* =13(*x − y*) + 43(2*x* + *y*)

*x* + *y* = *−*13(*x − y*) + 23(2*x* + *y*)

*x − y* = (3*x* + *y*) *−* 2(*x* + *y*)

2*x* + *y* =12(3*x* + *y*) + 12(*x* + *y*)

Note : *AX* = 0 is called a homogeneous system.

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Elementary row operations

Consider a matrix *A* = [*Aij*], where *Aij ∈ F,* a field.

The *ith* row of A is *Ri* = [*Ai*1*, Ai*2*, . . . , Ain*]

Type 1: Multiplication of one row of *A* by a non-zero scalar *c ∈ F* (*e* : *Ri ←− cRi*)

Type 2: Replacement of *ith* row by row *i* plus *c* times of row *j* where *c ∈ F* (*e* : *Ri ←− Ri* + *cRj*)

Type 3: Interchange of two rows (*e* : *Ri ←→ Rj*) 45

Inverse of the Type 1 elementary row operation

*A* =

"

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

Let *e* : *R*1 *←− cR*1*, c ̸*= 0

"

#

*e*(*A*) =

*cA*11 *cA*12 *cA*13 *A*21 *A*22 *A*23

We define *e*1 : *R*1 *←−* 1*cR*1 "

*e*1(*e*(*A*)) =

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

= *A*

Prove that *e*(*e*1(*A*)) = *A* =*⇒ e*1(*e*(*A*)) = *A* = *e*(*e*1(*A*)) 46

Inverse of the Type 1 elementary row operation

*A* =

"

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

Let *e* : *R*1 *←− cR*1*, c ̸*= 0

"

#

*e*(*A*) =

*cA*11 *cA*12 *cA*13 *A*21 *A*22 *A*23

We define *e*1 : *R*1 *←−* 1*cR*1 =*⇒ e*1 is the inverse elementary operation of *e*

*e*1(*e*(*A*)) =

"

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

= *A*

Prove that *e*(*e*1(*A*)) = *A* =*⇒ e*1(*e*(*A*)) = *A* = *e*(*e*1(*A*))47

Inverse of the Type 2 elementary row operation

*A* =

"

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

Let *e* : *R*2 *←− R*2 + *cR*1*, c ∈ F*

"

#

*e*(*A*) =

*A*11 *A*12 *A*13 *A*21 + *cA*11 *A*22 + *cA*12 *A*23 + *cA*13

We define *e*1 : *R*2 *←− R*2 *− cR*1. =*⇒ e*1(*e*(*A*)) = *A* Similarly, *e*(*e*1(*A*)) = *A* = *e*1(*e*(*A*))

Note: *e*1 is the inverse of *e*

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Inverse of the Type 3 elementary row operation

#

Let *e* : *R*1 *←→ R*2*.*

*A* =

"

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

*e*(*A*) =

"

*A*21 *A*22 *A*23 *A*11 *A*12 *A*13

#

We define *e*1 : *R*1 *←→ R*2*.* "

*e*1(*e*(*A*)) =

*A*11 *A*12 *A*13 *A*21 *A*22 *A*23

#

= *A*

Similarly, *e*(*e*1(*A*)) = *A* = *e*1(*e*(*A*))

Note: *e*1 is the inverse of *e* 49

Theorem 2

To each elementary row operation *e* there corresponds an elementary operation *e*1, of the same type as *e*, such that *e*(*e*1(*A*)) = *A* = *e*1(*e*(*A*)). In other words, inverse operation of an elementary operation exists and is of an elementary operation of the same type.

Proof :

| *Type* 1 | *Inverse of the Type* 1 |
| --- | --- |
| *e* : *Ri ←− cRi, c ̸*= 0 | *e*1 : *Ri ←−* 1*cRi* |

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Proof of Theorem 2 contd.

| *Type* 2 | *Inverse of the Type* 2 |
| --- | --- |
| *e* : *Ri ←− Ri* + *cRj* | *e*1 : *Ri ←− Ri − cRj* |

| *Type* 3 | *Inverse of the Type* 3 |
| --- | --- |
| *e* : *Ri ←→ Rj* | *e*1 : *Ri ←→ Rj* |

Note that for an *m × n* matrix *A*, *e*(*e*1(*A*)) = *A* = *e*1(*e*(*A*)) 51

Note

*e*1 : *R*2 *←−* 2*R*2 and *e*2 : *R*2 *←− R*2 *− R*1

*A* =

"

4 *−*1 5 2 1 7

#

*−→*

(*e*1)

"

4 *−*1 5 4 2 14

#

*−→*

(*e*2)

"

4 *−*1 5 0 3 9

#

= *B*

*e*2(*e*1(*A*)) = *B*

*e−*1

1: *R*2 *←−* 12*R*2 and *e−*1

2: *R*2 *←− R*2 + *R*1

*A* =

"

4 *−*1 5 2 1 7

#

*←−*

(*e−*1

1)

"

4 *−*1 5 4 2 14

#

*←−*

(*e−*1

2)

"

4 *−*1 5 0 3 9

#

= *B*

*e−*1

1(*e−*1

2(*B*)) = *A*

*A* and *B* are called row-equivalent matrices.52

Row-equivalent matrices

Definition : If *A* and *B* are *m × n* matrices over the field *F*, we say *B* is row-equivalent to *A* if *B* can be obtained from *A* by a finite sequence of elementary row operations.

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Theorem 3

If *A* and *B* are row-equivalent *m × n* matrices, the

homogeneous systems of linear equations *AX* = 0 and *BX* = 0 have exactly same solutions.

Proof: Suppose that we pass *A* to *B* by a finite sequence of elementary row operations :

*A* = *A*0 *−→ A*1 *−→ A*2 *−→ . . . −→ Ak* = *B*

Note: If

(1) *A*0*X* = 0 and *A*1*X* = 0 have same solutions,

(2) *A*1*X* = 0 and *A*2*X* = 0 have same solutions,

(j) *. . . , . . . ,*

(k) *Ak−*1*X* = 0 and *AkX* = 0 have same solutions,

then *AX* = 0 and *BX* = 0 have same solutions.

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Proof of Theorem 3 contd.

It is enough to prove that *AjX* = 0 and *Aj*+1*X* = 0 have exactly the same solutions(that is one elementary row operation doesn’t disturb the set of solutions).

Suppose that *B* is obtained from *A* by a single elementary row operation, say *e* (*i.e., e*(*A*) = *B*). No matter which of the types the operation is : (1) , (2) or (3), each equation in the system *BX* = 0 is a linear combination of the equations in *AX* = 0. Since *e−*1is an elementary row operation (*i.e., e−*1(*B*) = *A*), each equation in the system *AX* = 0 will also be a linear combination of equations in *BX* = 0. Hence these two systems are equivalent, and by Theorem 1, they have the same solutions.

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Problem 1

Show that the following systems are row-equivalent.

| *AX* = 0 | *BX* = 0 |
| --- | --- |
| 2*x*1 *− x*2 + 3*x*3 + 2*x*4 = 0 *x*1 + 4*x*2 *− x*4 = 0  2*x*1 + 6*x*2 *− x*3 + 5*x*4 = 0 | *x*3 *−*113*x*4 = 0 *x*1 +173*x*4 = 0 *x*2 *−*53*x*4 = 0 |

Solution at page number 8(Hoffman and Kunz)

It’s an assignment.

Note that solving the second system is easy !

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Note

Let us consider the matrix *B* from the previous problem.

*B* =



0 0 1 *−*113

1 0 0 1730 1 0 *−*53

 

• Note that the first non-zero entry of each non-zero row of *B* is 1.

• Note that each column of *B* which contains the leading non-zero entry of some row has all its other entries 0.

*B* =



0 0 1 *−*113

1 0 01730 1 0 *−*53





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Row-reduced matrix

An *m × n* matrix *R* is called row-reduced if :

(a) the first non-zero entry of each non-zero row of *R* is 1; (b) each column of *R* which contains the leading non-zero entry of some row has all its other entries 0.

Examples : (i) Identity matrix and (ii)the matrix *B* (previous problem)

Examples of non row-reduced matrices

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1 0 0 0

0 1 *−*1 0 0 0 1 0

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0 2 1

1 0 *−*3 0 0 0

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Problem 1

Find all solutions of the following system of equations by row-reducing the coefficient matrix.

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3*x*1 + 2*x*2 *−* 6*x*3 = 0

*−*4*x*1 + 5*x*3 = 0

*−*3*x*1 + 6*x*2 *−* 13*x*3 = 0

*−*73*x*1 + 2*x*2 *−*83*x*3 = 0

Solution: The coefficient matrix of the system is

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32 *−*6 *−*4 0 5 *−*3 6 *−*13 *−*732 *−*83

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Problem 1 contd.

*R*1 *←−* 3*R*1, *R*4 *←−* 3*R*4 *∼*

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1 6 *−*18

*−*4 0 5 *−*3 6 *−*13 *−*7 6 *−*8

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*R*2 *←− R*2 + 4*R*1, *R*3 *←− R*3 + 3*R*1, *R*4 *←− R*4 + 7*R*1

*∼*

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1 6 *−*18

0 24 *−*67 0 24 *−*67 0 48 *−*134

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Problem 1 contd.

*R*2 *←−* 124*R*2

*∼*

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0 1 *−*67240 24 *−*67 0 48 *−*134

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*R*1 *←− R*1 *−* 6*R*2, *R*3 *←− R*3 *−* 24*R*2, *R*4 *←− R*4 *−* 48*R*2

*∼*

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1 0 *−*54

0 1 *−*67240 0 0

0 0 0

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Thus

*x*1 *−*54*x*3 = 0

*x*2 *−*67~~24~~ *x*3 = 0 61

Problem 1 contd.

Let *x*3 = *a*. =*⇒ x*1 =54*a, x*2 =67~~24~~ *a*

Solution set, *S* = *{*(54*a,*67~~24~~ *a, a*) : *a ∈* R*}*

Note that

(i) *x*3 is a called free variable and

(ii) *x*1*, x*2 are called pivot variables.

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Theorem 4

Every *m × n* matrix over the field *F* is row-equivalent to a row-reduced matrix.

Proof : (Assignment)

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Problem 2

Find all solutions of the systems of linear equations *AX* = 2*X* and *AX* = 3*X* where

*A* =

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6 *−*4 0

4 *−*2 0 *−*1 0 3

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Solution : (i) The system *AX* = 2*X* is

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6 *−*4 0

4 *−*2 0 *−*1 0 3

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*xy z*

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= 2

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*xy z*

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=*⇒*

6*x −* 4*y* = 2*x* 4*x −* 2*y* = 2*y −x* + 3*z* = 2*z*

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Problem 2 contd.

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The coefficient matrix is 

4*x −* 4*y* = 0 4*x −* 4*y* = 0 *−x* + *z* = 0

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4 *−*4 0

4 *−*4 0 *−*1 0 1

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Let us find a row-reduced matrix which is row-equivalent to the above matrix. *R*3 *←−* (*−*1)*R*3, *R*3 *←→ R*1

*∼*

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1 0 *−*1

4 *−*4 0 4 *−*4 0

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Problem 2 contd.

*R*2 *←− R*2 *−* 4*R*1 and*R*3 *←−R*3*−*4*R*1 

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*R*2 *←− −*14*R*2 *R*3 *←−R*3 + 4*R*2

*∼ ∼*

1 0*−*1

0*−*4 4 0*−*4 4

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1 0*−*1

0 1*−*1 0*−*4 4

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1 0*−*1

0 1*−*1 0 0 0

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Problem 2 contd.

The equivalent system is

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*x − z* = 0

*y − z* = 0

Let *z* = *a* (Note that *z* is a free variable). Thus *x* = *a* = *y* (ii) Find all solutions of *AX* = 3*X*

The solution set is

*S* =*X ∈* R3: *AX* = 3*X*= *{*(0*,* 0*, a*) : *a ∈* R*}*.

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Note

*A* =

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0 1 *−*3 0 0 1

0 0 0 1 0 2 0 0 0 0 1 3 0 0 0 0 0 0

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1 *A* is a row-reduced matrix.

2 All non-zero rows are above zero rows.

3 The *ki* denotes the column which contains leading one (called pivot elements) (if exists) of *Ri* (row i).

*k*1 = 2, *k*2 = 4, and *k*3 = 5.

Note that *k*1 *< k*2 *< k*3.

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Row-reduced echelon matrix

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*A* =

0 0 0 1 0 2 0 0 0 0 1 3 0 0 0 0 0 0

blue zeros forms a staircase (echelon) from right to left. 69

Row-reduced echelon matrix

An *m × n* matrix *R* is called a row-reduced echelon matrix if: (a) *R* is row-reduced ;

(b) every row of *R* which has all its entries 0 occurs below every row which has a non-zero entry;

(c) if rows 1*,* 2*, . . . ,r* are the non-zero rows of *R*, and if the leading non-zero entry of row *i* occurs in column *ki*,

*i* = 1*,* 2*, . . . ,r*, then *k*1 *< k*2 *< . . . < kr*.

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*B* is a row-reduced matrix, but not a row-reduced echelon matrix

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Why?

*B* =

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0 0 1 *−*113

1 0 0 1730 1 0 *−*530 0 0 0

*k*1 = 3*, k*2 = 1*, k*3 = 2 which violates the condition (*c*) Could you find a a row-reduced echelon matrix *C* which is row-equivalent to *B*? (*R*1 *←→ R*2*, R*2 *←→ R*3)

*C* =

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1 0 0 173

0 1 0 *−*530 0 1 *−*1130 0 0 0

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Theorem 5

Every *m × n* matrix *A* is row-equivalent to a row-reduced echelon matrix.

Proof.

Assignment.

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Problem 3

Solve the system of linear equations *AX* = *b* where

*A* =

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1 1 *−*1 1 1 7 *−*5 *−*1

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and *b* =

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12 3

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Note 1:

Consider a row-reduced echelon matrix *R* and the system *RX* = 0 , where

*R* =

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0 1 *−*3 0 12

0 0 0 1 2 0 0 0 0 0

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*x*2 *−* 3*x*3 +12*x*5 = 0 *x*4 + 2*x*5 = 0

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No. of non-zero rows of *R, r* = 2, No. of variables, *n* = 5 *k*1 = 2*, k*2 = 4 =*⇒* Pivot variables = *{xk*1*, xk*2*}* = *{x*2*, x*4*}*. No. of free variables = *n − r* = 5 *−* 2 = 3,

Free variables = *{u*1*, u*2*, u*3*}* = *{x*1*, x*3*, x*5*}*.

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Note 1 contd.

*x*2 *−* 3*x*3 +12*x*5 = 0

*x*4 + 2*x*5 = 0

Set the free variables as :

*u*1 = *x*1 = *a, u*2 = *x*3 = *b, u*3 = *x*5 = *c*

=*⇒ x*2 = 3*b −*12*c, x*4 = *−*2*c*

Solution set *S* =(*a,* 3*b −*12*c, b, −*2*c, c*) : *a, b, c ∈* R 75

Observations from Note 1

*x*2 *−* 3*x*3 +12*x*5 = 0 *x*4 + 2*x*5 = 0

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=*⇒*

*xk*1 +X*n−r j*=1

*xk*2 +X*n−r j*=1

*C*1*juj* = 0 *C*2*juj* = 0

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Note 2

Consider an *m × n* row-reduced echelon matrix *R* with *r* non-zero rows. Let rows 1*,* 2*, . . . ,r* be the non-zero rows of *R*, and suppose that the leading non-zero entry of row *i* occurs in column *ki*. The system *RX* = 0 has *r* (non-trivial) equations. Let *xki*s are the pivot variables. Let *u*1*, u*2*, . . . , un−r* denote the (*n − r*) unknowns which are different from *xk*1*, xk*2*, . . . , xkr*. Then *r* non-trivial equations of *RX* = 0 are of the form

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Note 2 contd.

*xk*1 +X*n−r*

*j*=1

*xk*2 +X*n−r*

*j*=1

*C*1*juj* = 0 *C*2*juj* = 0

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*xkr* +X*n−r j*=1

*Crjuj* = 0

All the solutions of the system of equations *RX* = 0 are obtained by assigning any value whatsoever to *u*1*, u*2*, . . . , un−r*, and then computing the corresponding values of

*xk*1*, xk*2*, . . . , xkr*.

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Remarks (Note 2 contd.)

Thus, we have

(i) If *n > r*, then the system *RX* = 0 has at least one free variable and thus it has a non-trivial solution.

(ii) If *n* = *r*, then the system *RX* = 0 has no free variable and thus it has only trivial solution.

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